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A note on instanton counting for $\mathcal{N}=2$ gauge theories with classical gauge groups

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Abstract: We study the prepotential of $\mathcal{N}=2$ gauge theories using the instanton counting techniques introduced by Nekrasov. For the SO theories without matter we find a closed expression for the full prepotential and its string theory gravitational corrections. For the more subtle case of Sp theories without matter we discuss general features and compute the prepotential up to instanton number three. We also briefly discuss SU theories with matter in the symmetric and antisymmetric representations. We check all our results against the predictions of the corresponding Seiberg-Witten geometries.

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Contents

1. Introduction

The celebrated Seiberg-Witten solution [1] of $\mathcal{N} = 2$ gauge theories has been studied in great detail, but until recently no tractable method was known for obtaining the instanton expansion of the prepotential directly from many-instanton calculus. The final stumbling blocks were overcome in [2], building on previous work by several authors (see [3] for a review of many-instanton calculus predating [2]). Instanton corrections to the prepotential are determined by an integral over the moduli space of instantons. The crucial idea in [2] was to use localization techniques in a clever way to show that the integral over the moduli space of instantons can be computed from contributions of isolated fixed points, or equivalently, can be recast as a contour integral. This and related contour integrals had actually made an appearance much earlier [4], but the precise connection between these contour integrals and the prepotential was established only in [2]. For the gauge theories discussed in [2] the contour integral was explicitly evaluated and the result written as a sum over partitions. One surprising fact is that this result also encodes information about all the higher genus gravitational corrections which appear when the gauge theory is embedded in type II string theory. The results for these gravitational corrections were tested using topological strings in [5, 6]. Further work inspired by these instanton counting techniques can be found in [7]–[14].

The calculations in [2] were done for the gauge group SU(N). In this note we also consider the other classical gauge groups, SO(N) and Sp($2N$). For the $\mathcal{N}=2$ SO(N) gauge theory without matter we find the complete solution to the instanton counting problem, and write an explicit formula for the prepotential and its gravitational corrections as a sum over partitions, as was done in [2] for SU(N). For the Sp($2N$) theory, we write down the appropriate contour integral and evaluate it explicitly up to instanton number three. The structure of the poles for the Sp integral turns out to be much more complicated than for the cases of SU and SO gauge groups. We find the locations of all poles of the integrand, but have not been able to determine in closed form which ones are picked out by the contour prescription for arbitrary instanton numbers.

For the SU(N) theory with one hypermultiplet in the symmetric representation we also find the complete solution to the instanton counting problem in terms of a sum over partitions. On the other hand, for the SU(N) theory with one hypermultiplet in the antisymmetric representation we encounter problems similar to the ones that occur in the Sp($2N$) theory.

In retrospect, the fact that Sp($2N$) as well as SU(N) with antisymmetric matter are more complicated is not too surprising. Indeed, these theories are known to be more subtle already at the level of the Seiberg-Witten solution [15]–[17]. More recently, these theories were studied using the Dijkgraaf-Vafa matrix-model approach [18] and a modification of the original proposal was found to be required for these models [19, 20].

The next section 2 contains the analysis of the instanton counting for SO and Sp gauge groups. Explicit results for the prepotential up to instanton number 3 are listed in appendices A and B for SO and Sp, respectively. Section 3 presents the analysis for SU theories with matter hypermultiplets in the symmetric or antisymmetric representation, and explicit results up to instanton number 2 are presented in appendix C. In section 4 we list some open problems.

2. Instanton counting for $\mathcal{N}=2$ SO/Sp theories

In this section we discuss the $\mathcal{N}=2$ SO/Sp gauge theories. After briefly discussing the features common to both models we will discuss each case in turn.

2.1 ADHM data

A particularly expedient way of obtaining the ADHM instanton equations [21] for the $\mathcal{N}=2$ SU(N) gauge theory is from a system of k D(-1) branes (where k is the instanton number) and N D3-branes [22]. The (bosonic) ADHM data can be assembled into the four complex quantities (B_1, B_2, I, J) where B_1 and B_2 are in the adjoint of U(k), and I and J belong to the bifundamental representations $(\mathbf{k}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{k}}, \mathbf{N})$ of U(k)× SU(N), respectively. The U(k) gauge symmetry acts as

$$(B_a, I, J) \to (gB_a g^{-1}, gI, Jg^{-1})\,, \tag{2.1}$$

where $g \in \mathrm{U}(k)$. The ADHM equations are (see e.g. [7] for more details)

$$\begin{aligned}\mu_{\mathbb{R}} &= \left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = 0\,,\\ \mu_{\mathbb{C}} &= [B_1, B_2] + IJ = 0\,.\end{aligned} \tag{2.2}$$

The moduli space of instantons with instanton number k is the space of solutions to the above equations, modulo gauge transformations. As discussed in [2], it is actually convenient to consider a deformation of the first equation in (2.2) and take $\mu_{\mathbb{R}} \neq 0$. The space of solutions to (2.2) for $\mu_{\mathbb{R}} \neq 0$ (modulo gauge transformations) gives a resolution of singularities of the ADHM moduli space of instantons and can be regarded as the space of solutions to the second equation modulo *complex* gauge transformations, provided a stability condition is imposed (as usual in geometric invariant theory).

Notice that the linearization of the gauge symmetry action (2.1) gives a map

$$C : \mathbf{g} \to E \oplus E \oplus (V \otimes W^*) \oplus (V^* \otimes W)\,, \tag{2.3}$$

where $\mathbf{g}$ is the complexified Lie algebra of the instanton symmetry group $U(k)$, E is the representation space associated to the matrices B_a (in this case, since B_a belong to the adjoint representation, E is the complexified Lie algebra), V is the defining vector space of the instanton symmetry group, and W is the defining vector space of the gauge group. One also has the linearization of the second equation in (2.2), which gives a map

$$s : E \oplus E \oplus (V \otimes W^*) \oplus (V^* \otimes W) \to \mathbf{g}\,. \tag{2.4}$$

The maps (2.3), (2.4) fit together in the instanton deformation complex

$$\mathbf{g} \xrightarrow{C} E \oplus E \oplus (V \otimes W^*) \oplus (V^* \otimes W) \xrightarrow{s} \mathbf{g}\,. \tag{2.5}$$

The tangent space to the instanton moduli space is given locally by $\mathrm{Ker}\, s / \mathrm{Im}\, C$, see e.g. [3, 23] for more detailed discussions.

The ADHM data for SO/Sp gauge groups can be obtained by a projection of the SU data. The ADHM equations for SO/Sp were first obtained in [24].

In the D-brane language the projection is implemented by the addition of an orientifold (O3) plane to the D(-1)/D3 system. When placed on top of the stack of D3-branes the orientifold plane does not break any further supersymmetry. Depending on the charge of the orientifold one obtains either SO(N) or Sp($2N$) as the gauge group on the D3's. Due to the properties of the orientifold projection [25] one gets the "opposite" gauge group on the D(-1)'s i.e. Sp($2k$) and SO(k), respectively.

Implementing the projection on the ADHM data shows that for the SO(N) (Sp($2N$)) theory, the B_a's are *not* in the adjoint representation, but rather in the other two-index representation. For Sp($2k$) the adjoint is isomorphic to the symmetric representation whereas the B_a's are in the antisymmetric. For SO(k) the adjoint is isomorphic to the antisymmetric representation whereas the B_a's are in the symmetric. The orientifold projection on I and J relate them to their complex conjugates and thus halves the number of components.

The instanton deformation complex for SO/Sp has the same form as above (2.5), with the only difference that now E is the appropriate representation space. Notice that this description gives the correct number of parameters: the moduli space of instantons on $\mathbb{R}^4$ for a gauge group G has complex dimension $2kg^\vee$, where $g^\vee$ is the dual Coxeter number of the group. For $\mathrm{SO}(N)$ at instanton number k, I and J provide $2kN$ complex parameters, while $B_{1,2}$ are in the antisymmetric and give $2(k^2-k)$ complex parameters. The number of complex parameters modulo gauge transformations and the ADHM constraints is therefore $2k(N-2)$ from which it follows that the moduli space of instantons has complex dimension $2k(N-2)$. For $\mathrm{Sp}(2N)$, a similar counting gives $2k(N+1)$, both in agreement with the general formula for the dimension.

2.2 Instanton counting for $\mathrm{SO}(N)$

The instanton corrections to the Seiberg-Witten prepotential can be computed as integrals over the moduli space of instantons; see [3] for a review. These integrals are difficult to evaluate, but as shown in [2] one can use powerful localization techniques to simplify their computation.[1] The localization is done with respect to the group $\mathrm{U}(1)^N \times \mathrm{U}(1)^2$, where $\mathrm{U}(1)^N$ is the Cartan subgroup of the gauge symmetry, and $\mathrm{U}(1)^2$ is a global symmetry corresponding to an $\mathrm{SO}(2)\times\mathrm{SO}(2)$ rotation in spacetime, i.e. $\mathbb{R}^4$. This symmetry acts as follows on the ADHM fields: $(t_1,t_2)(B_1,B_2,I,J) = (t_1B_1, t_2B_2, I, t_1t_2J)$, where $(t_1,t_2) \in \mathrm{U}(1)\times\mathrm{U}(1)$. The ADHM equations (2.2) are unchanged under this action. It turns out that the fixed loci of this action are points, and this allows one to compute the integrals as sums over contributions coming from the fixed points.

Alternatively, one can start with the twisted version of the $\mathcal{N}=2$ theory written in terms of the ADHM fields and consider an equivariant extension of the BRST symmetry with respect to the above group action. Since the action is BRST exact, the path integral of the twisted theory can be calculated in the semiclassical approximation. By integrating out (B_1,B_2,I,J) (and their fermionic partners) one can reduce the path integral to a much simpler contour integral over the eigenvalues of a field ϕ which is part of the $\mathcal{N}=2$ topological multiplet (see [27, 8, 7] for details on this). From this contour integral, obtained from the path integral of the twisted theory, one can in the end extract the prepotential of the original $\mathcal{N}=2$ theory. It turns out that the poles that contribute to the contour integral are located precisely at the fixed points of the $\mathrm{U}(1)^N\times\mathrm{U}(1)^2$ action on the instanton moduli space, and the residue of the integral at each pole is precisely the contribution of the corresponding fixed point in the localization formula.

The analysis of [2] extends to the other classical gauge groups, and one can in fact write down the general form of the resulting contour integral.[2] As in the $\mathrm{U}(N)$ case, one has to consider an (equivariant) extension with respect to $H\times\mathrm{U}(1)^2$, where H is the Cartan subgroup of the gauge group. The field ϕ lives in the adjoint representation of the instanton symmetry group G_k at instanton number k, which is $\mathrm{U}(k)$, $\mathrm{Sp}(2k)$ and $\mathrm{SO}(k)$ for the gauge groups $\mathrm{U}(N)$, $\mathrm{SO}(N)$ and $\mathrm{Sp}(2N)$, respectively. After diagonalization ϕ can be written as

[1] See [26] for earlier ideas on using localization to evaluate the moduli space integrals.
[2] See [28] for the generalization to general groups of other aspects of [2, 9, 11].

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a vector $\phi = \sum_I \phi_I e_I$ in the root lattice of G_k, where e_I, $I = 1, \cdots, r_k$, is an orthonormal basis and r_k is the rank of G_k. Let $\alpha \in \Delta$ be the roots of the instanton symmetry group, and let $\mu \in \Lambda_B$ be the weights of the representation of the instanton symmetry group where the matrices B_a live. Finally, let $\hat{a}$ be a vector in the Cartan subalgebra $\mathbf{h}$ of the gauge group, and let $P(\phi_I) = \prod_i(\phi_I - \hat{a}_i)$. Notice that the $\hat{a}_i$'s play the role of equivariant parameters with respect to the symmetry action H. The integral then reads:

$$Z_k \propto \frac{1}{|\mathcal{W}_k|}\frac{1}{(2\pi i)^{r_k}} \oint \prod_I \frac{d\phi_I}{P(\phi_I)P(\phi_I+\epsilon)} \frac{\prod_{\alpha\in\Delta}(\phi\cdot\alpha)(\phi\cdot\alpha+\epsilon)}{\prod_{\mu\in\Lambda_B}(\phi\cdot\mu+\epsilon_1)(\phi\cdot\mu+\epsilon_2)}\,. \tag{2.6}$$

In this formula $\epsilon = \epsilon_1 + \epsilon_2$, where $\epsilon_{1,2}$ are the equivariant parameters in the Cartan subalgebra of the $\mathrm{SO}(2)\times\mathrm{SO}(2)$ rotation (in other words, $t_i = e^{\epsilon_i}$), and $|\mathcal{W}_k|$ is the order of the Weyl group of the instanton symmetry group. In (2.6) we have omitted an overall factor which depends on $\epsilon_{1,2}$. This integral has a nice geometric interpretation in terms of the instanton deformation complex associated to the ADHM equations (2.5). This complex has an equivariant extension with respect to the $H\times \mathrm{U}(1)^2$ action. Let Q be the defining representation space for $\mathrm{U}(1)^2$, and let W be the defining representation for H. Then, $(B_1, B_2) \in E\otimes Q$, $I \in V\otimes W^*$ and $J \in V^*\otimes W \otimes \wedge^2 Q$, and we obtain the equivariant complex:

$$\mathbf{g} \to (E\otimes Q)\oplus(V\otimes W^*)\oplus(V^*\otimes W\otimes\wedge^2 Q) \to \mathbf{g}\otimes\wedge^2 Q\,. \tag{2.7}$$

The integrand in (2.6) then computes the virtual Euler characteristic of the complex (2.7) [27, 2]. The denominator corresponds to the middle term, while the numerator corresponds to the first and third terms.

Let us now consider the $\mathcal{N}=2$ $\mathrm{SO}(N)$ super Yang-Mills theory. From the analysis of the ADHM data we know that the instanton symmetry group is $\mathrm{Sp}(2k)$ and that the fields B_a live in the antisymmetric representation of $\mathrm{Sp}(2k)$. The contour integral (2.6) reads in this case[3]

$$\begin{aligned} Z_k =\ & \frac{1}{2^k k!}\frac{\epsilon^k}{(\epsilon_1\epsilon_2)^k}\frac{1}{(2\pi i)^k}\oint \prod_{I=1}^k \mathrm{d}\phi_I \frac{(2\phi_I)^2((2\phi_I)^2-\epsilon^2)}{P(\phi_I)P(\phi_I+\epsilon)} \times \\ & \times \prod_{I<J} \frac{(\phi_I-\phi_J)^2((\phi_I-\phi_J)^2-\epsilon^2)(\phi_I+\phi_J)^2((\phi_I+\phi_J)^2-\epsilon^2)}{((\phi_I-\phi_J)^2-\epsilon_1^2)((\phi_I-\phi_J)^2-\epsilon_2^2)((\phi_I+\phi_J)^2-\epsilon_1^2)((\phi_I+\phi_J)^2-\epsilon_2^2)}\,. \end{aligned} \tag{2.8}$$

For $\mathrm{SO}(2N)$, $P(\phi) = \prod_{i=1}^N(\phi^2 - a_i^2)$ whereas for $\mathrm{SO}(2N+1)$, $P(\phi) = \phi\prod_{i=1}^N(\phi^2-a_i^2)$. Notice that the integrand in (2.8) is invariant under permutations of the ϕ_I's (which is a group of order $k!$) and also under the group $\mathbb{Z}_2^m$ generated by $a_i \to -a_i$, $i = 1, \cdots, m$. This is of course nothing but the Weyl group of $\mathrm{Sp}(2k)$.

The integral (2.8) can be evaluated by computing residues at the appropriate poles, as in [27, 2, 8]. In the case of the $\mathrm{SO}(2N)$ integral (2.8), the poles turn out to be essentially

[3] When integrating out J one seems to get $\sqrt{P(\phi_I+\epsilon)P(-\phi_I+\epsilon)}$ rather than $P(\phi_I+\epsilon)$. However as we exclusively set $\epsilon_1 = -\epsilon_2$ after evaluating the integral, this difference does not affect our results. When dealing with the $\epsilon \neq 0$ expressions this should be kept in mind. (Similar comments apply to the other contour integrals appearing in this paper.)

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the same as the ones for SU(N). However, as this is not completely obvious we will give some details. For SU(N) the location of the fixed points (and consequently the location of the poles) can be determined by solving the equations

$$\begin{aligned} &[\phi, B_1] = \epsilon_1 B_1\,, \qquad [\phi, B_2] = \epsilon_2 B_2\,, \qquad [B_1, B_2] + IJ = 0\,, \\ &\phi I - I\hat{a} = 0\,, \qquad J\phi - \hat{a}J - \epsilon J = 0\,, \end{aligned} \tag{2.9}$$

modulo complex gauge transformations. It is important to note that because of the specific choice of integration contour not all solutions to these equations are actually relevant to the evaluation of the integral. This is natural if we recall that the solutions to $[B_1, B_2]+IJ = 0$ only give solutions to the original ADHM equations (deformed to $\mu_{\mathbb{R}} \neq 0$) provided a certain stability condition is satisfied. It turns out that the poles that contribute to the contour integral are in one-to-one correspondence with the solutions of (2.9) that satisfy the stability condition. This condition can be phrased as follows. As discussed above $I \in V \otimes W^*$ where V and W are the representation spaces for the fundamental representation of U(k) and SU(N) respectively. The solutions which contribute to the integral are the ones which satisfy the condition that $B_1^n B_2^m I$ for $m, n = 1, 2, \ldots$ span the vector space V.

In [2] it was shown that the solution to the equations (2.9) subject to the stability condition are classified by Young tableaux in the following way. Each solution at instanton number k is given by a set of N Young tableaux $\mathbf{Y} = \{Y_1, \ldots, Y_N\}$ subject to the constraint that $\sum k_\ell = k$ where k_ℓ is the total number of boxes in the ℓth tableau. The boxes in Y_ℓ are labelled by pairs of integers (i_ℓ, j_ℓ), where i_ℓ labels the rows and j_ℓ the columns. In this language the fixed points are given by

$$\phi_{I_\ell} = a_\ell - (j_\ell - 1)\epsilon_1 - (i_\ell - 1)\epsilon_2\,. \tag{2.10}$$

We will now discuss how one can obtain the solution to the SO($2N$) problem by imposing a projection on the solution for SU($2N$). That one gets solutions by this projection is clear, but it is by no means a priori obvious that one gets all (relevant) solutions this way. The reason why there might be a problem is that one does not start with the most general solution to the SU problem, but instead only considers the solutions which satisfy the stability condition. In fact, as we will see later, for Sp($2N$) this naive method does not give all solutions. Nevertheless, the simplified method works for SO(N) and it is instructive to give some details. The methods we use (and the equations we solve) are similar to the ones used in [29] to derive all solutions to the classical vacuum equations for $N = 1^*$ theories with SO / Sp gauge groups.[4] After a change of basis to bring ϕ into diagonal form the Sp($2k$) conditions on the matrices ϕ, B_1 and B_2 can be written

$$\phi^T g = -g\phi\,, \quad B_1^T g = gB_1\,, \quad B_2^T g = gB_2\,, \quad g^T = -g\,, \quad gg^* = -\mathbb{1}_{2k}\,. \tag{2.11}$$

If, given a (stable) solution to the SU($2N$) problem, a $2k\times 2k$ matrix g can be found such that these equations are satisfied then the solution will descend to a solution of the SO($2N$) problem.

[4] In that paper the solutions were characterized in terms of SU(2) representations. One could also use this language here, but it is more convenient to work with the Young tableaux.

In matrix notation the solution to (2.9) for SU($2k$) schematically takes the block diagonal form

$$\phi = \mathrm{diag}(y_1, \ldots, y_{2k})\,, \qquad B_1 = \mathrm{diag}(x_1, \ldots, x_{2k})\,, \qquad B_2 = \mathrm{diag}(z_1, \ldots, z_{2k})\,. \tag{2.12}$$

Here y_ℓ is a diagonal matrix with elements $a_\ell - (i_\ell - 1)\epsilon_1 - (j_\ell - 1)\epsilon_2$ with some ordering for the Young tableau elements (i_ℓ and j_ℓ label rows and columns as already discussed), for instance left-to-right and down-up. Below we will use that there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry acting within *each* block,

$$\begin{aligned} \epsilon_1 &\to -\epsilon_1\,, & X_\ell &\to (X_\ell)^T\,, \\ \epsilon_2 &\to -\epsilon_2\,, & Z_\ell &\to (Z_\ell)^T\,. \end{aligned} \tag{2.13}$$

(This follows from the form of the equations $[\phi, B_a] = \epsilon_a B_a$.)

From the SO condition we get $a_{N+i} = -a_i$, $i = 1, \ldots, N$. The Sp($2k$) condition on ϕ can be written $(\phi_I + \phi_J) g_{IJ} = 0$ where $I, J = 1, \ldots, 2k$. In order for $g_{IJ} \neq 0$ to be possible $\phi_I + \phi_J$ must vanish. For generic a_ℓ, ϵ_1, and ϵ_2 this can only happen if ϕ_I is in block ℓ and ϕ_J is in block $\ell + N$ (since then $a_\ell + a_{\ell+N} = 0$). At first sight it seems that only for $i = j = 1$ in both the ℓth and the $(\ell + N)$th block is $\phi_I + \phi_J$ zero. This would contradict the $g^* g = -\mathbb{1}_{2k}$ condition and imply that there are no solutions. However, note that one can utilize the symmetry (2.13) to change the sign of $\epsilon_{1,2}$ in block $\ell + N$, $\ell = 1, \ldots, N$. After this change, the allowed g takes the form

$$g = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}, \tag{2.14}$$

where $D = \mathrm{diag}(D_1, \ldots, D_N)$ and each D_i is a diagonal matrix. It is also required that there is the same number of elements in blocks i and $i + N$ otherwise some of the D_i's would be rectangular matrices with at least one row (or column) full of zeroes. This would imply that $\det(g) = 0$ which contradicts $g^* g = -\mathbb{1}_{2k}$.

The above implies that the tableaux have to come in pairs; in the ϕ matrix one has to have $y_{i+N} = -y_i$. We are not quite finished since we also have to check that the Sp($2k$) conditions on $B_{1,2}$ can be satisfied. The Sp($2k$) condition on B_1 becomes (in matrix notation): $x_i D_i = D_i (x_{i+N})^T$ (and similarly for z_i). We can choose a basis in which all the non-zero elements in B_1 are normalized to 1, which implies $x_i = (x_{i+N})^T$ from which it can be shown that D_i is proportional to the unit matrix. The condition $g^* g = -\mathbb{1}_{2k}$ then implies that the proportionality constant is a phase, which can be removed by a unitary transformation. One also finds $z_i = (z_{i+N})^T$.

To summarize: we have shown that the (stable) solutions to (2.9), (2.11) (and hence the locations of the contributing poles) are classified by Young diagrams. Notice that only ϕ_I for $I = 1, \ldots, k$ enter in the integral. However, apart from the solution in (2.10) there are also contributing solutions that are obtained by permuting the k ϕ_I's, and we also have the possibility of choosing $\pm\phi_I$. This gives an overall factor $2^k k!$ which cancels the normalization $1/|\mathcal{W}_k|$ in (2.8).

For SO($2N + 1$) the only difference compared to SO($2N$) is that there is now an extra $\hat{a}_i$, $\hat{a}_0 \equiv 0$. The matrix ϕ still belongs to the adjoint of Sp($2k$) and the conditions (2.11) are unchanged. This means that there is an extra block that is independent of the a_i's. It not hard to show that a projection of the solution for $\mathrm{SU}(2N+1)$ would force this block to be filled with zeroes, but such solutions will always have moduli (giving rise to a vanishing integral) therefore this block has to be empty/absent. The solution is thus the same as for $\mathrm{SO}(2N)$. Note that even though the contributing fixed points are the same the residues are of course different.

Using a generalization of the method in [23, 2] one may explicitly evaluate the integral in terms of an infinite product. We will only write the expressions for $\epsilon_2 = -\epsilon_1 = \hbar$, which is enough to extract the Seiberg-Witten prepotential and the gravitational corrections. The computation can be easily done if we take into account that the residue at a given pole of the integral (2.6) can be translated into a localization computation: first, we compute the weights of the group action $H \times \mathrm{U}(1)^2$ on the different spaces appearing in (2.7). This is easily done by using the fact that (2.10) gives the weights associated to the defining representation of the instanton symmetry group. The residue is then given by the product of the weights associated to the first and third terms of (2.7), divided by the product of the weights associated to the middle term of (2.7).

For a specific partition $\mathbf{k} = (k_1, \ldots, k_N)$ one finds for $\mathrm{SO}(2N)$

$$Z_{\mathbf{k}} = 16^k \prod_{(i,j)\in Y_\ell} \left[a_\ell + \hbar\left(i - j - \frac{1}{2}\right)\right] [a_\ell + \hbar(i-j)]^2 \left[a_\ell + \hbar\left(i - j + \frac{1}{2}\right)\right] \times$$
$$\times \prod_{Y_\ell, Y_r} \prod_{m,n=1}^{\infty} \frac{[a_\ell - a_r + \hbar(k_{\ell,m} - k_{r,n} + n - m)][a_\ell + a_r + \hbar(k_{\ell,m} - \tilde{k}_{r,n} + n - m)]}{[a_\ell - a_r + \hbar(n-m)][a_\ell + a_r + \hbar(n-m)]}, \quad (2.15)$$

where it is assumed that the points where $(l, m) = (r, n)$ are excluded from the product. In (2.15) $k_{\ell,m}$ denotes the number of boxes in the mth row of the ℓth tableau and $\tilde{k}_{\ell,n}$ denotes the number of boxes in the nth column of the ℓth tableau. These definitions extend to all positive integers n, m: $k_{\ell,m}$ and $\tilde{k}_{\ell,n}$ are defined to be identically zero when n, m lie outside the tableau.

For $\mathrm{SO}(2N + 1)$ one finds the same product as for $\mathrm{SO}(2N)$ with the only difference that the product over single Young tableaux is instead given by

$$\prod_{(i,j)\in Y_\ell} \left[a_\ell + \hbar\left(i - j - \frac{1}{2}\right)\right] \left[a_\ell + \hbar\left(i - j + \frac{1}{2}\right)\right]. \quad (2.16)$$

The partition function for $\mathrm{SO}(N)$ at instanton number k, Z_k, is obtained by summing over all possible partitions $\mathbf{k}$ with k boxes in total, $Z_k = \sum_{\mathbf{k}} Z_{\mathbf{k}}$, where the sum is over $\mathbf{k}$ such that $\sum_{l,m} k_{l,m} = k$. Finally, the complete partition function is $Z = \sum_k L^k Z_k$ with $L = \Lambda^{2N-4}$. One has the important relation [2]

$$Z = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(a, \epsilon_1, \epsilon_2, \Lambda)\right], \quad (2.17)$$

where (for $\epsilon_2 = -\epsilon_1 = \hbar$)

$$\mathcal{F}(a, \hbar, \Lambda) = \sum_{g=0}^{\infty} \hbar^{2g} \mathcal{F}_g(a, \Lambda), \quad (2.18)$$

with

$$\mathcal{F}_g(a,\Lambda) = \sum_{k=1}^{\infty} L^k F_{g,k}(a)\,. \tag{2.19}$$

Here $\mathcal{F}_g(a,\Lambda)$ determines the genus g gravitational coupling, and $F_{g,k}$ is the contribution to this coupling at instanton level k. In particular, $\mathcal{F}_0$ is the Seiberg-Witten prepotential (excluding the perturbative part which will be discussed below).

We have checked that the above results lead to expressions for the first three instanton corrections to the prepotential which are in agreement with the results obtained using the Seiberg-Witten approach. Details about this check are included in appendix A. One can also check the results for the gravitational correction $\mathcal{F}_1$ against the general expression in terms of Seiberg-Witten data derived in [5, 30] (which is easily seen to extend to the $SO(N)$ case). This expression involves the determinant of the period matrix, whose instanton expansion in powers of Λ^{2N-2} is better calculated with the techniques of [31] after rewriting it in terms of hyperelliptic theta functions. Using this procedure we have checked the instanton counting result that $F_{1,1} = 0$ for $SO(N)$.

It is also of interest to connect the above discussion more directly to the corresponding Seiberg-Witten geometry. In [11] it was shown how to obtain the Seiberg-Witten data (i.e. the curve and the differential) from the instanton counting results for $\mathrm{SU}(N)$. (Another way to check the equivalence with the Seiberg-Witten approach was presented in [10]; see also [13].)

Here we will briefly discuss how the analysis in [11] is modified for the case of $\mathrm{SO}(N)$. Following [11] we introduce

$$\begin{aligned} f_{k_\ell}(x,\hbar) &= |x| + \sum_{i=1}^{\infty}\left[|x-\hbar(k_{\ell,i}-i+1)| - |x-\hbar(k_{\ell,i}-i)| + |x+\hbar(i-1) + |x+\hbar\, i|\right] \\ &= |x| + \sum_{j=1}^{\infty}\Big[|x+\hbar(\tilde{k}_{\ell,j}-j+1)| - |x+\hbar(\tilde{k}_{\ell,j}-j)| + |x-\hbar(j-1)| + |x-\hbar j|\Big]\,, \end{aligned} \tag{2.20}$$

as well as the function $\gamma_\hbar(x,\Lambda)$ satisfying

$$\gamma_\hbar(x+\hbar,\Lambda) + \gamma_\hbar(x-\hbar,\Lambda) - 2\gamma_\hbar(x,\Lambda) = \ln\left(\frac{x}{\Lambda}\right). \tag{2.21}$$

We furthermore define $f_{\mathbf{k},a}(x,\hbar) = \sum_\ell f_{k_\ell}(x-a_\ell)$. Using these definitions together with $\frac{\mathrm{d}^2}{\mathrm{d}x^2}|x| = 2\delta(x)$ one can show that

$$\begin{aligned} \mathcal{Z}_{\mathbf{k}} = \exp\Bigg\{&-\frac{1}{4}\fint f''_{\mathbf{k},a}(x) f''_{\mathbf{k},a}(y)\gamma_\hbar(x-y,\Lambda) - \frac{1}{4}\int f''_{\mathbf{k},a}(x) f''_{\mathbf{k},a}(y)\gamma_\hbar(x+y,\Lambda) + \\ &+\frac{1}{2}\int f''_{\mathbf{k},a}(x)\left[2\gamma_\hbar(x,\Lambda) + \gamma_\hbar\left(x+\frac{\hbar}{2},\Lambda\right) + \gamma_\hbar\left(x-\frac{\hbar}{2},\Lambda\right)\right]\Bigg\} \end{aligned} \tag{2.22}$$

is equal to $Z_{\mathrm{pert}} L^k Z_{\mathbf{k}}$ where $Z_{\mathbf{k}}$ was given in (2.15) and

$$\begin{aligned} Z_{\mathrm{pert}} = \exp\Bigg\{&-\sum_{\ell\neq r}\gamma_\hbar(a_\ell - a_r,\Lambda) - \sum_{\ell,r}\gamma_\hbar(a_\ell + a_r,\Lambda) + \\ &+\sum_\ell\left[2\gamma_\hbar(a_\ell,\Lambda) + \gamma_\hbar\left(a_\ell+\frac{\hbar}{2},\Lambda\right) + \gamma_\hbar\left(a_\ell-\frac{\hbar}{2},\Lambda\right)\right]\Bigg\}. \end{aligned} \tag{2.23}$$

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These expressions are valid for $\mathrm{SO}(2N)$; for $\mathrm{SO}(2N+1)$ the $2\gamma_\hbar(x,\Lambda)$ terms in (2.22) and (2.23) are absent.

Defining $\mathcal{F}^{\text{pert}}$ via $Z_{\text{pert}} = \exp[\mathcal{F}^{\text{pert}}/\hbar^2]$ and using [11, 13]

$$\hbar^2\gamma_\hbar(x,\Lambda) = \frac{x^2}{2}\ln\left(\frac{x}{\Lambda}\right) - \frac{3x^2}{4} + \mathcal{O}(\hbar^2) \equiv \gamma_0(x,\Lambda) + \mathcal{O}(\hbar^2)\,, \tag{2.24}$$

it is easy to check that $\mathcal{F}_0^{\text{pert}}$ agrees with the usual perturbative result for the prepotential

$$\mathcal{F}_0^{\text{pert}} = -\sum_{\alpha\in\Delta_+}(a\cdot\alpha)^2\ln\left(\frac{a\cdot\alpha}{\Lambda}\right) + \text{quadratic}\,, \tag{2.25}$$

where the sum is over the positive roots of $\mathrm{SO}(N)$. The higher order terms in $\mathcal{F}^{\text{pert}}$ give conjectural expressions for the perturbative part of the gravitational corrections.

More generally one can analyze the full partition function, $\mathcal{Z}$, obtained by summing over terms of the form (2.22) for all possible partitions at all instanton numbers, in the limit $\hbar \to 0$. It was argued in [11] that in this limit the sum goes over to an integral and $\mathcal{Z}$ can be obtained from the saddle-point of the action ($\mathcal{Z} = \exp[\mathcal{E}/\hbar^2]$)

$$\mathcal{E} = -\frac{1}{4}⨍ f''(x)f''(y)\gamma_0(x-y,\Lambda) - \frac{1}{4}\int f''(x)f''(y)\gamma_0(x+y,\Lambda) + 2\int f''(x)\gamma_0(x,\Lambda)\,, \tag{2.26}$$

where $f''(x)$ is a continuous (real) function localized on intervals around the a_i's and $\gamma_0(x,\Lambda)$ was defined in (2.24). Comparing this expression to the one in [11] we note the expected connection with the $\mathrm{SU}(2N)$ theory with 4 massless hypermultiplets in the fundamental representation. We also note that at the saddle point (2.26) is equal to the total prepotential (including the perturbative piece), $\mathcal{E} = \mathcal{F}_{0,\text{tot}}$. As usual in saddle-point problems of this kind it is convenient to introduce the resolvent

$$R(z) = \frac{1}{2}\int \mathrm{d}x\,\frac{f_a''(x)}{z-x}\,, \qquad R(x+i\varepsilon) - R(x-i\varepsilon) = \pi i f_a''(x)\,, \tag{2.27}$$

which satisfies

$$a_i = \frac{1}{2\pi i}\oint_{A_i} zR(z)\,\mathrm{d}z\,, \qquad \frac{1}{2\pi i}\oint_{A_i} R(z)\,\mathrm{d}z = 1\,, \tag{2.28}$$

where A_i is the contour surrounding the ith cut. It should also be straightforward to derive

$$\frac{\partial\mathcal{F}_{0,\text{tot}}}{\partial a_i} = \frac{1}{2\pi i}\oint_{B_i} zR(z)\,\mathrm{d}z\,, \qquad \frac{1}{2\pi i}\oint_{B_i} R(z)\,\mathrm{d}z = 0\,, \tag{2.29}$$

and discuss in more detail the form of the curve following [11] (see also [12]) but we will not do so here.

2.3 Instanton counting for $\mathrm{Sp}(2N)$

For the $\mathrm{Sp}(2N)$ case, ϕ belongs to the adjoint of $\mathrm{SO}(k)$ (where k is the instanton number) and we have to distinguish between odd and even instanton numbers. For $k = 2n+1$ we have (using a convenient normalization)

$$
\begin{aligned}
Z_{2n+1} = {} & \frac{(-1)^n}{2^{n+1}n!}\frac{\epsilon^n}{(\epsilon_1\epsilon_2)^{n+1}}\frac{1}{\sqrt{P(0)P(\epsilon)}}\frac{1}{(2\pi i)^n}\oint\prod_{I=1}^{n}\mathrm{d}\phi_I\frac{1}{P(\phi_I)P(\phi_I+\epsilon)}\times \\
& \times\frac{\phi_I^2(\phi_I^2-\epsilon^2)}{(\phi_I^2-\epsilon_1^2)((2\phi_I)^2-\epsilon_1^2)(\phi_I^2-\epsilon_2^2)((2\phi_I)^2-\epsilon_2^2)} \\
& \times\prod_{I<J}\frac{(\phi_I-\phi_J)^2((\phi_I-\phi_J)^2-\epsilon^2)(\phi_I+\phi_J)^2((\phi_I+\phi_J)^2-\epsilon^2)}{((\phi_I-\phi_J)^2-\epsilon_1^2)((\phi_I-\phi_J)^2-\epsilon_2^2)((\phi_I+\phi_J)^2-\epsilon_1^2)((\phi_I+\phi_J)^2-\epsilon_2^2)}
\end{aligned}
$$

whereas for $k=2n$ we instead get

$$
\begin{aligned}
Z_{2n} = {} & \frac{(-1)^n}{2^n n!}\frac{\epsilon^n}{(\epsilon_1\epsilon_2)^n}\frac{1}{(2\pi i)^n}\oint\prod_{I=1}^{n}\mathrm{d}\phi_I\frac{1}{P(\phi_I)P(\phi_I+\epsilon)}\frac{1}{((2\phi_I)^2-\epsilon_1^2)((2\phi_I)^2-\epsilon_2^2))}\times \\
& \times\prod_{I<J}\frac{(\phi_I-\phi_J)^2((\phi_I-\phi_J)^2-\epsilon^2)(\phi_I+\phi_J)^2((\phi_I+\phi_J)^2-\epsilon^2)}{((\phi_I-\phi_J)^2-\epsilon_1^2)((\phi_I-\phi_J)^2-\epsilon_2^2)((\phi_I+\phi_J)^2-\epsilon_1^2)((\phi_I+\phi_J)^2-\epsilon_2^2)}\,. \qquad (2.30)
\end{aligned}
$$

One can follow the same approach as for $\mathrm{SO}(N)$ and project Nekrasov's solution. However, this procedure does *not* give the right result: some solutions are missing. The problem is that some solutions to the fixed point equations for $\mathrm{SU}(2N)$, which do not satisfy the stability condition turn out to contribute (after projection) to the evaluation of the above integrals. This can be phrased as saying that the stability condition is different in the SU and Sp cases. The new stable solutions not obtained via a projection of Nekrasov's solution are all of the form $\phi_I = 0 + \mathcal{O}(\epsilon_1, \epsilon_2)$ i.e. localized near $\phi_I = 0$. These extra solutions are the manifestation in the present framework of the extra cut around $x = 0$ in the Seiberg-Witten curve (B.1) or equivalently of the "Sp(0)" factors [19, 20] in the Dijkgraaf-Vafa matrix model approach.

To get an understanding of the extra solutions, we should consider all solutions to the fixed point equations (2.9) together with the $\mathrm{SO}(k)$ projections

$$\phi^T f = -f\phi\,, \quad B_1^T f = fB_1\,, \quad B_2^T f = fB_2\,, \quad f^T = f\,, \quad ff^* = \mathbb{1}_k\,. \qquad (2.31)$$

If, given a solution to (2.9), we can find a $k\times k$ matrix f such that these conditions are satisfied then the solution to (2.9) descend to a solution of the $\mathrm{Sp}(2N)$ k-instanton problem. We have solved these equations from first principles using the techniques in [29]. Some of the solutions have moduli, i.e. undetermined parameters. These solutions do not contribute since they give a vanishing result for the path integral (see e.g. [27] for a discussion of this point in a similar context). Since the new solutions not obtained by projection do not depend on the a_i's we focus on the a_i-independent solutions (such solutions have $I = J = 0$). We have found that these solutions can be represented pictorially in terms of stacks of rows of boxes. Each box corresponds to an eigenvalue of ϕ. These boxes are placed in the xy-plane in a symmetric way. The rows are parallel to the x-axis and can be stacked on top of each other. The rules are:

- The diagram has to be symmetric under $x \to -x$ together with $y \to -y$.

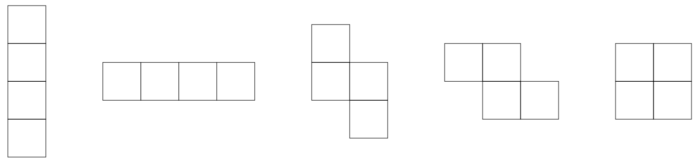

Figure 1: Connected diagrams relevant to the $k = 4$ calculation.

- A row on top of another (above the x-axis) can not extend to the right of the row below it.
- Rows with one box are special. Such rows can only be placed to the far left of the row below it.
- A row of length n_1 placed on top of a row of length n_2 is not allowed if $(n_2 - n_1)/2$ is equal to the distance between the right end of the lower row and the right end of the upper row.

In general one has several disconnected diagrams. But no two diagrams are allowed to be the same (this would give moduli). These rules give a_i-independent solutions to the fixed-point equations which do not have moduli. This is best illustrated by an example. The allowed connected diagrams with four boxes are those in figure 1.

These diagrams correspond to the solutions: $(\phi_1, \phi_2) = (\epsilon_2/2, 3\epsilon_2/2)$; $(\epsilon_1/2, 3\epsilon_1/2)$; $(-\epsilon_1/2, -\epsilon_1/2 + \epsilon_2)$; $(-\epsilon_1 + \epsilon_2/2, \epsilon_2/2)$, $(-\epsilon_1/2 + \epsilon_2/2, \epsilon_1/2 + \epsilon_2/2)$ (modulo the action of the Weyl group). Here (ϕ_1, ϕ_2) are the two eigenvalues which enter in the integral (the other two components encoded in the above diagrams are minus these by the SO projection). The Weyl group acts on these solutions and one would expect that the number of times each type of solution appears in the evaluation of the integral to be equal to an integer multiple of the number of elements in a Weyl orbit, but explicit calculations indicate that this is not the case.

From the above analysis we know all possible locations of the fixed points. The remaining problem is to determine which of these possibilities are actually realized in the evaluation of the integral and how many times each solution appears. In general, not all solutions are relevant since the choice of integral contour/pole prescription excludes some possibilities. (It turns out that all the diagrams in the above figure contribute in the evaluation of the $k = 4$ integral.) It is natural to expect that diagrams which do not look like two Young diagrams glued together (some with the first row (column) of half the height (width)) are to be excluded. There may be some connection between these diagrams with boxes of half the normal height (width) and the spinor representations of SO. In addition to the fact that some diagrams are excluded one also needs to determine how many times each of the diagrams that do contribute occur. Unfortunately we have been unable to solve this combinatorial problem, i.e. we have not been able to determine what the correct 'stability condition' is for the extra solutions appearing in the Sp$(2N)$ case.

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However, even without an explicit product formula one may still check that the above integrals lead to expressions which agree with previous results. In appendix B we check that the above integrals give rise to expressions for the first three instanton corrections to the prepotential which agree with the ones obtained using the Seiberg-Witten approach.

3. Instanton counting for $\mathrm{SU}(N)$ with $\Box\!\Box/\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ matter

Other examples which can be treated using the methods of [2] include $\mathcal{N}=2$ $\mathrm{SU}(N)$ gauge theories with matter in two-index representations, i.e. $\Box\!\Box$ (symmetric) or $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ (antisymmetric). In this section we briefly discuss these two cases.

3.1 $\mathrm{SU}(N)$ with matter in the $\Box\!\Box$ representation

Going through the same steps as for the previously discussed cases one may derive the contour integral

$$
\begin{aligned}
Z_k = {} & \frac{1}{k!}\frac{\epsilon^k}{(\epsilon_1\epsilon_2)^k}\frac{1}{(2\pi i)^k}\oint\prod_{I=1}^{k}\mathrm{d}\phi_I\frac{(2\phi_I+\epsilon_1)(2\phi_I+\epsilon_2)\prod_{i=1}^{N}(\phi_I+a_i)}{P(\phi_I)P(\phi_I+\epsilon)}\times \\
& \times\prod_{I<J}\frac{(\phi_I-\phi_J)^2((\phi_I-\phi_J)^2-\epsilon^2)(\phi_I+\phi_J+\epsilon_1)(\phi_I+\phi_J+\epsilon_2)}{((\phi_I-\phi_J)^2-\epsilon_1^2)((\phi_I-\phi_J)^2-\epsilon_2^2)(\phi_I+\phi_J)(\phi_I+\phi_J+\epsilon)}\,. \qquad (3.1)
\end{aligned}
$$

This case was briefly mentioned in [32].[5] From the integral (3.1) it is easy to see (assuming that the contour is the same as in the case without matter) that the solutions which contribute are the same ones as in the pure $\mathrm{SU}(N)$ theory. Using this result one may derive

$$
\begin{aligned}
Z_{\mathbf{k}} = {} & 4^k\prod_{(i,j)\in Y_\ell}\left[a_\ell+\hbar\left(i-j-\frac{1}{2}\right)\right]^{1/2}[a_\ell+\hbar(i-j)]\left[a_\ell+\hbar\left(i-j+\frac{1}{2}\right)\right]^{1/2}\times \\
& \times\prod_{Y_\ell,Y_r}\prod_{m,n=1}^{\infty}\frac{[a_\ell-a_r+\hbar(k_{\ell,m}-k_{r,n}+n-m)][a_\ell+a_r+\hbar(n-m)]^{1/2}}{[a_\ell-a_r+\hbar(n-m)][a_\ell+a_r+\hbar(k_{\ell,m}-\tilde{k}_{r,n}+n-m)]^{1/2}}\,, \qquad (3.2)
\end{aligned}
$$

where $\hbar=\epsilon_2=-\epsilon_1$. In this expression it is assumed that the points $(\ell,m)=(r,n)$ are excluded.

We have checked that the above expressions agree (up to two instantons) with the results obtained in [33] (see appendix C for details).

It should also be possible to analyze this model along the lines of [11] and in particular derive the cubic curve (C.1). As a first step we write down the analogue of (2.22):

$$
\begin{aligned}
\mathcal{Z}_{\mathbf{k}} = \exp\Bigg\{ & -\frac{1}{4}\fint f''_{\mathbf{k},a}(x)f''_{\mathbf{k},a}(y)\gamma_\hbar(x-y,\Lambda)+\frac{1}{8}\int f''_{\mathbf{k},a}(x)f''_{\mathbf{k},a}(y)\gamma_\hbar(x+y,\Lambda)+ \\
& +\frac{1}{4}\int f''_{\mathbf{k},a}(x)\left[2\gamma_\hbar(x,\Lambda)+\gamma_\hbar\left(x+\frac{\hbar}{2},\Lambda\right)+\gamma_\hbar\left(x-\frac{\hbar}{2},\Lambda\right)\right]\Bigg\} \qquad (3.3)
\end{aligned}
$$

[5]There is a typo in the formulas given in this reference. The unpublished corrected version agrees with our expression. We thank S. Naculich for correspondence on this point.

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which leads to the following analogue of (2.23)

$$Z_{\text{pert}} = \exp\Bigg\{ -\sum_{\ell\neq r} \gamma_\hbar(a_\ell - a_r, \Lambda) + \frac{1}{2}\sum_{\ell,r} \gamma_\hbar(a_\ell + a_r, \Lambda) + \\ + \frac{1}{2}\sum_{\ell} \left[2\gamma_\hbar(a_\ell, \Lambda) + \gamma_\hbar\left(a_\ell + \frac{\hbar}{2}, \Lambda\right) + \gamma_\hbar\left(a_\ell - \frac{\hbar}{2}, \Lambda\right)\right]\Bigg\} \,. \tag{3.4}$$

From which one can extract, using (2.24),

$$\mathcal{F}_0^{\text{pert}} = -\sum_{\alpha\in\Delta_+} (a\cdot\alpha)^2 \ln\left(\frac{a\cdot\alpha}{\Lambda}\right) + \frac{1}{2}\sum_{\mu\in R_W} (a\cdot\mu)^2 \ln\left(\frac{a\cdot\mu}{\Lambda}\right) + \text{quadratic}\,, \tag{3.5}$$

where Δ_+ is the set of positive roots of $\mathrm{SU}(N)$ and R_W is the set of weights of the symmetric representation. This result agrees with the known result.

3.2 SU(N) with matter in the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation

For this case one finds the contour integral

$$Z_k = \frac{1}{k!}\frac{\epsilon^k}{(\epsilon_1\epsilon_2)^k}\frac{1}{(2\pi i)^k}\oint \prod_{I=1}^{k} \mathrm{d}\phi_I \frac{\prod_{i=1}^N(\phi_I + a_i)}{P(\phi_I)P(\phi_I+\epsilon)(2\phi_I)(2\phi_I+\epsilon)} \times \\ \times \prod_{I<J} \frac{(\phi_I-\phi_J)^2((\phi_I-\phi_J)^2-\epsilon^2)(\phi_I+\phi_J+\epsilon_1)(\phi_I+\phi_J+\epsilon_2)}{((\phi_I-\phi_J)^2-\epsilon_1^2)((\phi_I-\phi_J)^2-\epsilon_2^2)(\phi_I+\phi_J)(\phi_I+\phi_J+\epsilon)}\,. \tag{3.6}$$

For this model one encounters the problem that in addition to the fixed points of the pure $\mathrm{SU}(N)$ theory extra solutions to the fixed point equations need to be taken into account to get agreement with previous results. This is similar to the situation that occurred for $\mathrm{Sp}(2N)$, which should come as no surprise given the similarity of the Seiberg-Witten curves and the results in [19, 20]. We have been unable to find a closed expression which includes the extra contributions. Instead we have checked that the above integral leads to results (up to two-instanton order) which are in agreement with the results [33] obtained using the Seiberg-Witten procedure.

4. Discussion

Clearly there are several issues one would like to understand better. One problem is to determine the correct stability condition for the $\mathrm{Sp}(2N)$ and $\mathrm{SU}(N)+\begin{smallmatrix}\square\\\square\end{smallmatrix}$ cases and use it to explicitly evaluate the contour integrals by summing up the residues.

Another problem concerns the relation with topological strings. In the case of $\mathrm{SO}(2N)$ theories, one can in principle compute the prepotential of the 5d theory on $\mathbb{R}^4\times\mathbf{S}^1$, following the same steps as in [2]. The resulting expression might be related to a topological string amplitude on the Calabi-Yau obtained by fibering a D_N singularity over a $\mathbb{P}^1$ base. This would be very interesting since no results are known for such manifolds, but unfortunately there seems to be a conundrum: one would expect it to be possible to write the topological string amplitude in terms of the Kähler parameters of the Calabi-Yau. These are presumably in a one-to-one correspondence with the simple roots of D_N, but the five-dimensional prepotential seems to involve combinations of the Kähler parameters that do not correspond to sums of simple roots with positive coefficients. It would be interesting to clarify this and in that way obtain predictions for the topological string amplitudes.

The methods we used in this paper can also be applied to a similar problem, the determination of the so called bulk (or principal) contribution to the Witten index in SYM quantum mechanics. It is known that this quantity can be written as a so called Yang-Mills integral which can be reduced to a contour integral using methods similar to the ones discussed in this paper (see e.g. [27, 34] and references therein for further details). Explicit expressions for the bulk part of the Witten index are known only for the case of SU(N) (see e.g. [27] and references therein). However for the other classical gauge groups much less is known and the results in the literature have been obtained order-by-order using computer assisted calculations [35]. The bulk Witten index can be studied for SYM quantum mechanics obtained by dimensional reduction of $d = 4$, $d = 6$ and $d = 10$ supersymmetric Yang-Mills theories. Our analysis corresponds most closely to the $d = 6$ case. As an example, for the $d = 6$ Sp($2N$) case the fixed points can be obtained by solving

$$\begin{aligned} [\phi, B_1] &= \epsilon_1 B_1 , & [\phi, B_2] &= \epsilon_2 B_2 , & [B_1, B_2] &= 0 , \\ \phi^T g &= -g\,\phi , & B_1^T g &= -g B_1 , & B_2^T g &= -g B_2 , \\ g^T &= -g , & g\,g^* &= -\mathbb{1}_{2N} . \end{aligned} \tag{4.1}$$

These equations are similar to the ones we solved in section 2.3 and again the solution can be represented pictorially in terms of stacks of row of boxes. In addition to the rules we listed in section 2.3 we now have the additional restrictions (related to the fact that the equations are not quite the same):

- Only rows containing an even number of boxes can be placed on top of the x-axis. For example, diagrams containing only one row can only have an even number of boxes.
- Only an odd number of boxes are allowed to touch each other along the x-axis.

These rules (together with the ones in section 2.3) give the solutions to the fixed-point equations which do not have moduli. Thus we have determined the possible locations of poles in the integrand of the contour integral which calculates the bulk part of the Witten index for the Sp($2N$) quantum mechanics arising from $d = 6$. However, just as in section 2.3 not all of these solutions are relevant to the evaluation of the integral; what is lacking is an understanding of the stability condition. It is our hope that our results will be of some help in resolving the longstanding problem of calculating the bulk Witten index for SO/Sp supersymmetric quantum mechanics.

One can also apply our methods to cases other than $d = 6$. For instance, for the Sp($2N$) (SO(N)) theory arising from $d = 4$ the solutions to the fixed point equations are given by direct sums of distinct even-dimensional (odd-dimensional) representations of SU(2), but once again the correct stability condition is not known.

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A. SO(N) instanton corrections up to $k = 3$

Here we discuss the explicit expressions which result from the formulæ in section 2.2 and check that they agree with known results for $k = 1, 2, 3$. The contribution of each fixed point is obtained by evaluating the residue of the integrand in (2.8) at the poles (2.10). Since the poles are as in the U(N) case, the calculation is essentially the same as in that case. The result for $\epsilon_1 = -\epsilon_2 = \hbar$ can be written in terms of the function

$$U_l(x) = \frac{1}{\prod_{m \neq l}((a_l - a_m)^2 + x)^2} \frac{16(a_l + x)^4}{\prod_m (a_l + a_m + x)^2} \tag{A.1}$$

for SO($2N$), and

$$U_l(x) = \frac{1}{\prod_{m \neq l}((a_l - a_m)^2 + x)^2} \frac{16(a_l + x)^2}{\prod_m (a_l + a_m + x)^2} \tag{A.2}$$

for SO($2N+1$). Below we will use the notation $U_l = U_l(0)$, $U_l^{(n)} = (\partial U_l(x)/\partial x)_{x=0}$. After a simple (but tedious) calculation one obtains the first few instanton corrections in the SO(N) case:

$$\begin{aligned}
\hbar^2 Z_1 &= \sum_l U_l \\
\hbar^4 Z_2 &= \frac{1}{4}\sum_{l=1}^{m} \Big[U_l U_l(\hbar) g_l^{(1)}(\hbar) + U_l U_l(-\hbar) g_l^{(1)}(-\hbar)\Big] + \frac{1}{2}\sum_{l\neq m} U_l U_m g(a_l - a_m) g(a_l + a_m) \\
\hbar^6 Z_3 &= \frac{1}{36}\sum_l U_l \Big[U_l(\hbar)U_l(2\hbar) g_l^{(2)}(\hbar) + U_l(-\hbar)U_l(-2\hbar) g_l^{(2)}(-\hbar) + \\
&\qquad\qquad + 4U_l(\hbar)U_l(-\hbar) g^{(3)}(\hbar)\Big] + \\
&\quad + \frac{1}{4}\sum_{l\neq m} U_l U_m \Big[U_l(\hbar) m(\hbar, a_l - a_m) m(\hbar, a_l + a_m) g^{(1)}(\hbar) + \\
&\qquad\qquad + U_l(-\hbar) m(-\hbar, a_l - a_m) m(-\hbar, a_l + a_m) g^{(1)}(-\hbar)\Big] + \\
&\quad + \frac{1}{6}\sum_{l\neq m\neq n} U_l U_m U_n g(a_l - a_m) g(a_l - a_n) g(a_m - a_n) g(a_l + a_m) g(a_l + a_n) g(a_m + a_n)
\end{aligned} \tag{A.3}$$

where we have used the notation

$$m(x,y) = \frac{1}{\left[1 - \frac{2x^2}{y(x+y)}\right]^2}, \qquad g(y) = \frac{1}{\left(1 - \frac{\hbar^2}{y^2}\right)^2}, \qquad g_l^{(1)}(x) = \frac{1}{\left[1 - (\frac{x}{2a_l + x})^2\right]^2},$$

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$$
\begin{aligned}
g_l^{(2)}(x) &= \frac{1}{\left[1-(\frac{x}{2a_l+x})^2\right]^2\left[1-(\frac{x}{2a_l+2x})^2\right]^2\left[1-(\frac{x}{2a_l+3x})^2\right]^2}\,, \\
g_l^{(3)}(x) &= \frac{1}{\left[1-(\frac{x}{2a_l+x})^2\right]^2\left[1-(\frac{x}{2a_l-x})^2\right]^2\left[1-(\frac{x}{2a_l})^2\right]^2}\,.
\end{aligned} \tag{A.4}
$$

From these equations one can compute the instanton corrections to the prepotential, $F_k \equiv F_{0,k}$ for $k = 1, 2, 3$. The results are as follows:

$$
\begin{aligned}
F_1 &= \sum_l U_l\,, \\
F_2 &= \frac{1}{4}\sum_l U_l\left(U_l'' + \frac{U_l}{a_l^2}\right) + \sum_{l\neq m} U_l U_m\left(\frac{1}{(a_l-a_m)^2} + \frac{1}{(a_l+a_m)^2}\right), \\
F_3 &= \frac{1}{36}\sum_l U_l\left[U_l U_l^{(4)} + 2U_l U_l''' + 3(U_l'')^2\right] + \frac{1}{16}\sum_l \frac{1}{a_l^4}\left(5U_l^3 - 4a_l U_l^2 U_l' + 4U_l^2 U_l''\right) + \\
&\quad + \sum_{l\neq m} U_l U_m\Bigg\{5U_l\left(\frac{1}{(a_l-a_m)^4} + \frac{1}{(a_l+a_m)^4}\right) - 2U_l'\left(\frac{1}{(a_l-a_m)^3} + \frac{1}{(a_l+a_m)^3}\right) \\
&\qquad\qquad + U_l''\left(\frac{1}{(a_l-a_m)^2} + \frac{1}{(a_l+a_m)^2}\right)\Bigg\} + \\
&\quad + \sum_{l\neq m} U_l^2 U_m\left\{\frac{1}{a_l^2}\left(\frac{1}{(a_l-a_m)^2} + \frac{1}{(a_l+a_m)^2}\right) + \frac{4}{(a_l-a_m)^2(a_l+a_m)^2}\right\} + \\
&\quad + 2\sum_{l\neq m\neq n} U_l U_m U_n\Bigg\{\frac{1}{(a_l-a_m)^2(a_m-a_n)^2} + \frac{1}{(a_l+a_m)^2(a_m+a_n)^2} \\
&\qquad\qquad + 2\frac{1}{(a_l-a_m)^2(a_n+a_m)^2}\Bigg\}\,.
\end{aligned} \tag{A.5}
$$

We can compare these results to the ones obtained from the Seiberg-Witten approach by utilizing the relation between the prepotential of the SO($2N$) theory and that of the SU($2N$) theory with 4 massless flavors [15]. Using this relation together with the SU results in [36] (or [2]) one finds agreement with our results above after a rescaling $F_k \to 2^{-4k+1}F_k$.

B. Sp($2N$): instanton corrections up to $k = 3$

Here we will perform a check of our results by comparing the first three instanton corrections to the prepotential obtained using the Seiberg-Witten approach to those obtained using many-instanton counting.

Seiberg-Witten approach. The Seiberg-Witten curve for the Sp($2N$) gauge theory without matter is (see e.g. [15, 37] and references therein)

$$
y^2 + 2\,y\left[x^2\prod_{i=1}^{N}(x^2-e_i^2) + L\right] + L^2 = 0\,, \tag{B.1}
$$

where $L = \Lambda^{2N+2}$. The quantum order parameters are [38]

$$a_i = e_i + \sum_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} \frac{(-1)^n L^{n+2m}}{2^{2m}(m!)^2 n!}\left[\left(\frac{\partial}{\partial x}\right)^{2m+n-1} R_k(x)^n S_k(x)^m\right]_{x=e_k}, \tag{B.2}$$

where $S_k(x) \equiv (R_k(x))^2$ and

$$\frac{1}{x^2\prod_{i=1}^N (x^2-e_i^2)} = \frac{R_k(x)}{x-e_k} = \frac{R_0(x)}{x^2}. \tag{B.3}$$

For later reference we have also introduced $R_0(x)$ with $S_0(x) \equiv (R_0(x))^2$. The instanton expansion of the prepotential can be obtained using the recursive methods developed in [36], which leads to (after using various identities)[6]

$$\begin{aligned} -F_1 &= R_0(0)\,, \\ -F_2 &= -\frac{1}{2}\sum_k S_k(a_k) - \frac{1}{8}S_0''(0)\,, \\ -F_3 &= R_0(0)\left(\frac{3}{2}\sum_k \frac{S_k(a_k)}{a_k^2} + \frac{1}{96}S_0^{(4)}(0)\right). \end{aligned} \tag{B.4}$$

Many-instanton counting. From the contour integrals in section 2.3 we find

$$\begin{aligned} \hbar^2 Z_1 &= \frac{1}{2P(0)}\,, \\ \hbar^4 Z_2 &= \frac{1}{8}\left[\frac{1}{P(\hbar/2)^2} + \frac{\hbar^2}{8}\sum_k \frac{1}{(a_k^2-(\hbar/2)^2)^2 a_k^2 \prod_{j\neq k}(a_k^2-a_j^2)^2}\right], \\ \hbar^6 Z_3 &= \frac{1}{16}\frac{1}{P(0)}\left[\frac{2}{9}\frac{1}{P(\hbar)^2} + \frac{1}{9}\frac{1}{P(\hbar/2)^2} + \right. \\ &\qquad \left. + \frac{\hbar^2}{8}\sum_k \frac{a_k^2}{(a_k^2-\hbar^2)^2(a_k^2-(\hbar/2)^2)^2\prod_{j\neq k}(a_k^2-a_j^2)^2}\right]. \end{aligned} \tag{B.5}$$

Using the relations

$$F_1 = \hbar^2 Z_1\,, \qquad F_2 = \hbar^2\left(Z_2 - \frac{Z_1^2}{2}\right), \qquad F_3 = \hbar^2\left(Z_3 - Z_1 Z_2 + \frac{Z_1^3}{3}\right), \tag{B.6}$$

and extracting the $\hbar$-independent pieces we find agreement with the above results (B.4) obtained from the Seiberg-Witten procedure provided that $F_k^{\rm here} = -\frac{F_k^{\rm there}}{k\,2^k}$. Part of this difference can be removed by a rescaling of Λ; the other part is a result of different conventions.

As a further check we have also verified that the above results are consistent with the ones for $\mathrm{SO}(2N+1)$ using the isomorphism $\mathrm{SO}(5) \cong \mathrm{Sp}(4)$ and allowing for a rescaling of the Λ's in the two theories. (The translation between the order parameters, d_i, of SO(5) and the ones of Sp(4), a_i, are: $d_1 = a_1 - a_2$ and $d_2 = a_1 + a_2$.)

[6]The first two expressions were first obtained in [17].

C. SU(N)+ $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$/$\square\square$: instanton corrections up to $k=2$

Here we will perform a check of our results for the SU(N) + $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$/$\square\square$ theories by comparing the first two instanton corrections to the prepotential obtained using the Seiberg-Witten approach to those obtained from many-instanton counting.

C.1 SU(N) + $\square\square$

Seiberg-Witten approach. The cubic Seiberg-Witten curve for the SU(N) gauge theory with one matter hypermultiplet in the symmetric ($\square\square$) representation is given by [37]

$$y^3 + P(x)y^2 + x^2 P(-x)L + x^6 L^3 = 0\,, \tag{C.1}$$

where $P(x) = \prod_{i=1}^{N}(x - e_i)$ and $L = \Lambda^{N-2}$. The instanton expansion of the prepotential has been obtained to the first few orders in [33].[7] After using various identities one finds the expressions

$$\begin{aligned} -F_1 &= -\sum_k S_k(a_k)\,, \\ -F_2 &= \frac{1}{4}\sum_k S_k(a_k)S_k''(a_k) + \sum_{k\neq l}\frac{S_k(a_k)S_l(a_l)}{(a_l-a_k)^2} - \frac{1}{2}\sum_{k,l}\frac{S_k(a_k)S_l(a_l)}{(a_l+a_k)^2}\,, \end{aligned} \tag{C.2}$$

where we have used the definition

$$S_k(x) = \frac{x^2\prod_i(-x-a_i)}{\prod_{j\neq k}(x-a_j)^2}\,. \tag{C.3}$$

Many-instanton counting. From the expressions in section 3.1 we find

$$\begin{aligned} \hbar^2 Z_1 &= -4\sum_k\left(a_k^2 - \left(\frac{\hbar}{2}\right)^2\right)\frac{\prod_j(a_k+a_j)}{\prod_{j\neq k}(a_k-a_j)^2} \\ \hbar^4 Z_2 &= 16\Bigg[\frac{1}{4}\sum_k\left(a_k - 3\frac{\hbar}{2}\right)a_k\,(a_k-\hbar)\left(a_k+\frac{\hbar}{2}\right)\frac{\prod_j(a_k-\hbar+a_j)(a_k+a_j)}{\prod_{j\neq k}(a_k-\hbar-a_j)^2(a_k-a_j)^2} + \\ &\quad + \frac{1}{4}\sum_k\left(a_k + 3\frac{\hbar}{2}\right)a_k(a_k+\hbar)\left(a_k-\frac{\hbar}{2}\right)\frac{\prod_j(a_k+\hbar+a_j)(a_k+a_j)}{\prod_{j\neq k}(a_k+\hbar-a_j)^2(a_k-a_j)^2} + \\ &\quad + \frac{1}{2}\sum_{k\neq l}\frac{(a_k^2-(\hbar/2)^2)(a_l^2-(\hbar/2)^2)(a_k-a_l)^4((a_k+a_l)^2-\hbar^2)\prod_j(a_k+a_j)(a_l+a_j)}{((a_l-a_k)^2-\hbar^2)^2(a_k+a_k)^2\prod_{j\neq k}(a_k-a_j)^2\prod_{j\neq l}(a_l-a_j)^2}\Bigg]. \end{aligned} \tag{C.4}$$

Using (B.6) and extracting the leading pieces, we find agreement with the results in (C.2) provided that we identify $F_k^{\text{here}} = (-4)^k(-1)^{Nk}F_k^{\text{there}}$.

[7] The one-instanton expression was first obtained in [16].

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C.2 $\mathrm{SU}(N) + \begin{smallmatrix}\square\\\square\end{smallmatrix}$

Seiberg-Witten approach. The non-hyperelliptic Seiberg-Witten curve for the $\mathrm{SU}(N)$ gauge theory with one matter hypermultiplet in the antisymmetric ($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) representation is given by [37]

$$y^3 + \left[P(x) + \frac{3L}{x^2}\right] y^2 + \frac{L}{x^2}\left[P(-x) + \frac{3L}{x^2}\right] y + \frac{L^3}{x^6} = 0\,, \tag{C.5}$$

where $P(x) = \prod_{i=1}^{N}(x - e_i)$ and $L = \Lambda^{N+2}$. The instanton expansion of the prepotential has been obtained to the first few orders in [33].[8]

After using various identities one obtains

$$\begin{aligned}
F_1 &= \sum_k S_k(a_k) - 2S_0(0)\,,\\
F_2 &= \frac{1}{4}\sum_k S_k(a_k)S_k''(a_k) + \sum_{k\neq l}\frac{S_k(a_k)S_l(a_l)}{(a_l - a_k)^2} - \frac{1}{2}\sum_{k,l}\frac{S_k(a_k)S_l(a_l)}{(a_l + a_k)^2} -\\
&\quad - 2S_0(0)\sum_k \frac{S_k(a_k)}{a_k^2} + \frac{1}{2}S_0(0)S_0''(0) - \frac{1}{2}S_0'(0)S_0'(0)\,,
\end{aligned} \tag{C.6}$$

where we have used the definitions

$$\frac{\prod_k(-x - a_k)}{x^2\prod_k(x - a_k)^2} \equiv \frac{S_k(x)}{(x - a_k)^2} \equiv \frac{S_0(x)}{x^2}\,. \tag{C.7}$$

Many-instanton counting. Evaluation of the contour integrals gives

$$\begin{aligned}
\hbar^2 Z_1 &= -\tfrac{1}{4}\sum_k \frac{\prod_j(a_k + a_j)}{a_k^2\prod_{j\neq k}(a_k - a_j)^2} - \frac{1}{2}\frac{\prod_i a_i}{\prod_i(-a_i^2)}\,,\\
\hbar^4 Z_2 &= \frac{1}{16}\Bigg[\frac{1}{4}\sum_k \frac{1}{(a_k - \hbar)\,a_k\,(a_k - \hbar/2)^2}\frac{\prod_j(a_k - \hbar + a_j)(a_k + a_j)}{\prod_{j\neq k}(a_k - \hbar - a_j)^2(a_k - a_j)^2} +\\
&\quad + \frac{1}{4}\sum_k \frac{1}{(a_k + \hbar)a_k(a_k + \hbar/2)^2}\frac{\prod_j(a_k + \hbar + a_j)(a_k + a_j)}{\prod_{j\neq k}(a_k + \hbar - a_j)^2(a_k - a_j)^2} +\\
&\quad + \frac{1}{2}\sum_{k\neq l}\frac{(a_k - a_l)^4((a_k + a_l)^2 - \hbar^2)\prod_j(a_k + a_j)(a_l + a_j)}{((a_l - a_k)^2 - \hbar^2)^2(a_k + a_k)^2\,a_l^2\,a_k^2\prod_{j\neq k}(a_k - a_j)^2\prod_{j\neq l}(a_l - a_j)^2} -\\
&\quad - 2\sum_k \frac{a_k^2}{a_k^2 - \hbar^2}\frac{\prod_j(a_j)(a_k + a_j)}{\prod_j(-a_j)^2\prod_{i\neq k}(a_k - a_i)^2} + 2\frac{1}{\prod_j(\hbar/2 - a_j)(-\hbar/2 - a_j))}\Bigg]\,,
\end{aligned} \tag{C.8}$$

Using (B.6) and extracting the leading pieces we find agreement with (C.6) provided that we identify $F_k^{\text{here}} = (-4)^{-k}(-1)^{Nk}F_k^{\text{there}}$.

To arrive at the result (C.8) one needs to choose a contour which picks out a very particular set of poles. We do not understand why this particular contour should be chosen (knowledge of the correct stability condition should shed light on this).

[8] The one-instanton expression was first obtained in [39].

As a consistency check we have checked that the above results are consistent with the fact that for SU(3) the antisymmetric representation is isomorphic with the fundamental representation.[9]

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